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Self-similar convergence of a shock wave in a heat conducting gas[☆]

S.V. Khabirov

Ufa, Russia

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ABSTRACT

The problem of the convergence of a spherical shock wave (SW) to the centre, taking into account the thermal conductivity of the gas in front of the SW, is considered within the limits of a proposed approximate model of a heat conducting gas with an infinitely high thermal conductivity and a small temperature gradient, such that the heat flux is finite in a small region in front of the converging SW. In this model, there is a phase transition in the surface of the SW from a perfect gas to another gas with different constant specific heat and the heat outflow. The gas is polytropic and perfect behind the SW. Constraints are derived which are imposed on the self-similarity indices as a function of the adiabatic exponents on the two sides of the SW. In front of the SW, the temperature and density increase without limit. In the general case, a set of self-similar solutions with two self-similarity indices exists but, in the case of strong SW close to the limiting compression, there are two solutions, each of which is completely determined by the motion of the spherical piston causing the self-similar convergence of the SW.

It has been shown in the problem of the self-similar convergence of a shock wave (SW) in a quiescent perfect gas that the SW intensity increases without limit and obeys a definite law of convergence for certain adiabatic exponents.[1–14] There may be or may not be self-similar convergence for other adiabatic exponents or the solution is not unique. The choice of the perfect gas model is explained by the fact that the specific heat and viscosity give rise to rapidly attenuating perturbations near the SW.

An approximate model of the differential equations and a detonation wave has been proposed.[15] Nine type of shock adiabatics have been listed as a function of the arbitrary supply or removal of heat and, also, the change in the equation of state in the shock transition in the case of a steady non-point sink or source. Self-similar motion has not been considered.

1. Approximate model

A spherical shock wave converges to the centre. Suppose the gas thermal conductivity is high in the sphere in front of the SW and the temperature gradient is small, such that the heat flux is finite. Behind the SW, the gas is not heat conducting and there is no heat flux. The equation of state in the SW can be varied. This model was proposed by Nigmatulin for investigating the asymptotic behaviour of the convergence of a SW near the centre.

In front of the SW, the approximate model of a heat conducting gas is represented by the hyperbolic system of equations of motion of a perfect polytropic gas[15]

$$\begin{aligned}
&\rho_1(U_{1t}+U_1U_{1r})+R_1T_1\rho_{1r}=0\\
&\rho_{1t}+U_1\rho_{1r}+\rho_1\left(U_{1r}+2r^{-1}U_1\right)=0\\
&q_{1r}+2r^{-1}q_1+c_{\upsilon 1}\rho_1\left(T_{1t}+T_1(\gamma_1-1)\left(U_{1r}+2r^{-1}U_1\right)\right)=0
\end{aligned}\qquad(1.1)$$

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E-mail address: habirov@anrb.ru.

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Here, U_1 is the radial velocity, $T_1 = T_1(t)$ is the temperature, ρ_1 is the density, q_1 is the heat flux, and $c_{\upsilon 1}$ and c_{p1} are the specific heats. Using the formulae

$$p_1 = R_1\rho_1 T_1, \quad R_1 = c_{p1} - c_{\upsilon 1}, \quad \varepsilon_1 = c_{\upsilon 1}T_1, \quad \gamma_1 = c_{p1}c_{\upsilon 1}^{-1}$$

$$a_1 = \sqrt{\gamma_1 R_1 T_1}, \quad S_1 = c_{\upsilon 1}\ln T_1 - R_1 \ln\rho_1 + \mathrm{const}$$

the pressure, gas constant, internal energy, adiabatic exponent, speed of sound and the entropy can be calculated.

Behind the SW, all quantities are written with the subscript 2. The equations of gas dynamics are (Ref. [6], §15)

$$\begin{aligned}&\rho_2(U_{2t} + U_2 U_{2r}) + R_2(\rho_2 T_{2r} + T_2\rho_{2r}) = 0\\ &\rho_{2t} + U_2\rho_{2r} + \rho_2\left(U_{2r} + 2r^{-1}U_2\right) = 0\\ &T_{2t} + U_2 T_{2r} + (\gamma_2 - 1)T_2(U_{2r} + 2r^{-1}U_2) = 0\end{aligned} \tag{1.2}$$

The relations

$$\begin{aligned}&m \equiv \rho_1(U_1 - D) = \rho_2(U_2 - D)\\ &R_1\rho_1 T_1 + m(U_1 - D) = R_2\rho_2 T_2 + m(U_2 - D)\\ &m\left((U_1 - D)^2 + 2c_{p1}T_1\right) + 2q_1 = m\left((U_2 - D)^2 + 2c_{p2}T_2\right)\end{aligned} \tag{1.3}$$

are satisfied (Ref. [14], §2) in a SW with an equation of motion $r_\upsilon = D(t, t_\upsilon) < 0$.

A value of $q_1 > 0$ corresponds to the supply of heat to the SW and a value $q_1 < 0$ corresponds to the removal of heat from the SW.

Eqs. (1.1)–(1.3) allow of the expansions (Ref. [7], §19)

$$Y_1 = t\partial_t + r\partial_r$$

$$Y_2 = r\partial_r + U_1\partial_{U_1} + 2T_1\partial_{T_1} + 3q_1\partial_{q_1} + D\partial_D + U_2\partial_{U_2} + 2T_2\partial_{T_2}$$

$$Y_3 = \rho_1\partial_{\rho_1} + q_1\partial_{q_1} + \rho_2\partial_{\rho_2}$$

The Abelian algebra of operators generates a two-parameter family of one-dimensional subalgebras $Y = Y_1 + \alpha Y_2 + \beta Y_3$ to which the self-similar solutions correspond (Ref. [1], Chapter 4, §1; Ref. [7], §8). The parameters α and β are the self-similarity indices. The convergence of the SW to the centre begins at the instant $t = 0$. We will consider the motion of the gas when $t < 0$. The invariants of the operator Y give a representation of the solution.

2. The self-similar solution in front of the shock wave

In the sphere, bounded by the SW, a representation of the solution is

$$U_1 = -|t|^\alpha u_1(\xi), \quad \rho_1 = |t|^\beta\lambda_1(\xi), \quad q_1 = |t|^{3\alpha+\beta}\sigma(\xi), \quad T_1 = KR_1^{-1}|t|^{2\alpha} \tag{2.1}$$

where

$$\xi = r|t|^{-\alpha-1}, \quad 0 \le \xi \le \xi_0, \quad u_1 > 0, \quad \lambda_1 > 0, \quad -1 < \alpha < 0$$

and K is a positive constant. The equality $\xi = \xi_0$ determines the motion of the front of the invariant SW. The case is considered when $\sigma < 0$ (removal of heat from the SW).

For a fixed $t < 0$ and $r \to 0$, the focusing conditions

$$u_1(0) = 0, \quad \sigma(0) = 0$$

are satisfied.

Substitution of expression (2.1) into system (1.1) gives the system of ordinary differential equations

$$\begin{aligned}&\frac{d\sigma}{d\eta} + \frac{2\sigma}{\eta} = Q(\eta) \Rightarrow \sigma = \eta^{-2}\int_0^\eta \zeta^2 Q(\zeta)d\zeta\\ &\frac{d\lambda_1}{d\eta} = \lambda_1\Lambda(\eta) \Rightarrow \lambda_1 = L\exp\int_0^\eta \Lambda(\zeta)d\zeta, \quad L > 0\\ &\frac{du}{d\eta} = \frac{-\alpha\eta u^2 + \alpha(\alpha+1)\eta^2 u + 2u + \beta\eta}{\eta[(u - (\alpha+1)\eta)^2 - 1]}\end{aligned} \tag{2.2}$$

where

$$u_1 = uK^{1/2},\quad \xi = \eta K^{1/2},\quad Q(\eta) = K^{3/2}\lambda_1(\eta)\left(\frac{2\alpha}{\gamma_1 - 1} + u' - \frac{2u}{\eta}\right)$$

$$\Lambda(\eta) = \frac{-2u^2 + \eta u(3\alpha - \beta + 2) + \beta(\alpha + 1)\eta^2}{\eta\left[(u - (\alpha + 1)\eta)^2 - 1\right]}$$

The last equation of (2.2) is invariant under the reflection $\eta \to -\eta$, $u \to -u$, it has an integral line $\eta = 0$ and it can have five singular points.

The curvilinear separatrix of the saddle point occurring at the point (0,0) when $\beta \neq 0$ solves the problem. It has asymptotic behaviour when $\eta \to 0$: $u \sim -\beta\eta/3$. Since $u > 0$, $\eta > 0$, then

$$\beta < 0 \tag{2.3}$$

The asymptotic behaviour of the other gas dynamic functions when $\xi \to 0$ is

$$u_1 \sim -\frac{1}{3}\beta\xi,\quad \lambda_1 \sim L\left(1 - \frac{1}{6K}\left(\beta + \frac{1}{3}\right)\xi^2\right),\quad \sigma \sim \frac{1}{3}LK\left(\frac{2\alpha}{\gamma_1 - 1} - \beta\right)\xi \tag{2.4}$$

Since $\sigma \le 0$, then

$$2\alpha - (\gamma_1 - 1)\beta \le 0 \tag{2.5}$$

When $\alpha = 0$ and $\gamma_1 \neq 1$, we have $\sigma > 0$, that is, heat goes to the wave. When $\alpha = 0$ and $\gamma_1 = 1$, we have $q_1 = 0$. In both cases there is a contradiction.

The lines $\beta = 0$ and $2\alpha - (\gamma_1 - 1)\beta = 0$ limit the domain of possible values of the self-similarity parameters.

When $\beta = 0$, Eq. (2.2) has two integral lines: $\eta = 0$ and $u = 0$. There are three singular points in the domain $\eta \ge 0$, $u \ge 0$. The point $O(0,0)$ is a saddle point with the separatrices $\eta = 0$, $u = 0$. The point $Q(\eta, u) = ((1 + \alpha)^{-1}, 0)$, when $\alpha < -2/3$, is a node of the tangent $u = 0$ of the integral curves and with a separate whisker, the tangent to which is the line $(\alpha + 1)(2u + \alpha\eta) = \alpha$. When $\alpha > -2/3$, this point is a saddle point with a separatrix, the tangent to which is the line $(\alpha + 1)(2u + \alpha\eta) = \alpha$ and, when $\alpha = -2/3$, it is a saddle point - node while, in the domain $u > 0$, this point is a saddle point with a separatrix, the tangent to which is the line $\eta = 3(u + 1)$.

Hence, when $\beta = 0$, $-2/3 \le \alpha < 0$, there is a unique integral curve which passes into the origin of coordinates through the point Q. The third singular point $B(\eta, u) = (-2/\alpha, -3-2/\alpha)$ is a saddle point when $-2/3 < \alpha < 0$.

The solution has the following asymptotic behaviour when $\eta \to (1 + \alpha)^{-1}$:

$$\begin{aligned}
&u \sim -\frac{1}{2}\alpha\left(\eta - \frac{1}{1+\alpha}\right),\quad \lambda_1 \sim \frac{1}{2}L\left(2 + \frac{\alpha}{1+\alpha} - \alpha\eta\right)\\
&\sigma \sim \frac{\alpha}{\gamma_1 - 1}LK^{3/2}\left[\frac{1}{12(1+\alpha)^4}\left(\gamma_1 - 1 - \frac{1}{4}\alpha(3\gamma_1 - 1)\right)\eta^{-2}\right.\\
&\left.+\frac{1}{12}\left(2(\gamma_1 + 3) - \alpha\frac{5 - \gamma_1}{1+\alpha}\right)\eta - \frac{1}{16}(4(\gamma_1 - 1) + \alpha(5\gamma_1 - 9))\eta^2\right]
\end{aligned} \tag{2.6}$$

3. The self-similar solution behind the shock wave

Behind the SW, the representation of the solution is

$$U_2 = rt^{-1}u_2(\xi),\quad \rho_2 = |t|^\beta\lambda_2(\xi),\quad T_2 = R_2^{-1}r^2t^{-2}\tau(\xi);\quad \xi = r|t|^{-\alpha-1} \tag{3.1}$$

For a fixed t, the magnitude of r varies from $\xi_0|t|^{\alpha+1}$ to ∞, and this means that $\xi_0 \le \xi < \infty$. In the case of fixed r, $\alpha > -1$ and $t \to 0$, we have $\xi \to \infty$ (the instance of convergence of the SW). Since the magnitudes of U_2 and T_2 are bounded for fixed $r \neq 0$, then

$$u_2 \sim \xi^{-1/(\alpha+1)},\quad \tau \sim \xi^{-2/(\alpha+1)},\quad \lambda_2 \sim \xi^{\beta/(\alpha+1)}\quad \text{when}\quad \xi \to \infty$$

The conditions for the SW convergence are

$$u_2 > 0,\quad \lambda_2 > 0,\quad \tau > 0$$

Substitution of expressions (3.1) into Eq. (1.2) gives a system of ordinary differential equations (Ref. [1], Ch. 4., §2)

$$\begin{aligned}
&(\alpha + 1 - u_2)\xi\lambda_2^{-1}\lambda_2' = 3u_2 + \beta + kl^{-1}\\
&(\alpha + 1 - u_2)\xi\tau^{-1}\tau' = (\gamma_2 - 1)kl^{-1} + (3\gamma_2 - 1)u_2 - 2\\
&l\xi u_2' = k
\end{aligned} \tag{3.2}$$

where

$$l = (u_2 - \alpha - 1)^2 - \gamma_2\tau,\quad k = u_2(1 - u_2)(u_2 - \alpha - 1) + \tau(3\gamma_2 u_2 + 2\alpha + \beta)$$

The magnitude of ξ becomes larger as r increases. The velocity modulus $|U_2|$ decreases behind the front and, for fixed $t=t_0$, the magnitude of $u_2=U_2t_0r^{-1}\to 0$ when $r\to\infty$. This means that $0\le u_2\le u_2(\xi_0)$ and $d\xi/du_2<0$ in the case of a solution which describes the SW convergence.

Integration of system (3.2) reduces to integration of the ordinary differential equation

$$\frac{d\tau}{du_2}=\frac{\tau}{1+\alpha-u_2}\left[\gamma_2-1+lk^{-1}((3\gamma_2-1)u_2-2)\right] \tag{3.3}$$

It has two integral lines: $\tau=0$ and $u_2=\alpha+1$ and can have six singular points for values of the parameters in the domain

$$-1<\alpha<0,\quad 1\le\gamma_2\le 2,\quad \beta\le 0,\quad 2\alpha\le(\gamma_1-1)\beta$$

The point $O(\tau=0,\ u_2=0)$, is a node with a detached whisker which has a tangent $(\alpha+1)u_2+(2\alpha+\beta)\tau=0$. The tangent $\tau=0$ is common to the remaining integral curves.

The existence of a saddle point $(1,0)$ and a complex singular point $(1+\alpha,0)$ in the integral curve in the (u_2,τ) plane contradicts the physical meaning of the problem of the SW convergence since the inequalities $0<u_2<1+\alpha$ must be satisfied. We shall therefore not consider the integral curves passing through these points.

The points $D_\pm$ satisfy the equalities $l=0$ and $k=0$ from which the equations

$$\gamma_2\tau=(u_2-\alpha-1)^2 \tag{3.4}$$

$$\gamma_2u_2(u_2-1)=(u_2-\alpha-1)(3\gamma_2u_2+2\alpha+\beta) \tag{3.5}$$

follow. Eq. (3.4) determines the parabola P, the points of which correspond to the sound characteristics of system (1.2), and Eq. (3.5) follows from the relation on the characteristic.[3]

The roots of quadratic equation (3.5) are real if the values of the parameters α and β lie outside the ellipse

$$\beta^2+2(\gamma_2+2)\beta\alpha+\left(9\gamma_2^2+4\gamma_2+4\right)\alpha^2+4\gamma_2(3\gamma_2+2)\alpha+4\gamma_2\beta+4\gamma_2^2=0 \tag{3.6}$$

The roots of Eq. (3.5) lie in the interval $0<u_2<1+\alpha$ if and only if

$$-1<\alpha<0 \tag{3.7}$$

For the singular point C, we have

$$u_2=\frac{2}{3\gamma_2-1},\quad \tau=\frac{6(\gamma_2-1)(3(\gamma_2-1)+\alpha(3\gamma_2-1))}{(3\gamma_2-1)^2(6\gamma_2+(3\gamma_2-1)(2\alpha+\beta))}$$

If the point C lies on the parabola P, then it coincides with one of the points D_+ or D_- (Fig. 1).

Inequalities (2.3), (2.5) and (3.7) bound the two domains D_1 and D_2 of possible values of the parameters α and β which lie outside the ellipse (3.6) (they are shown hatched in Fig. 2).

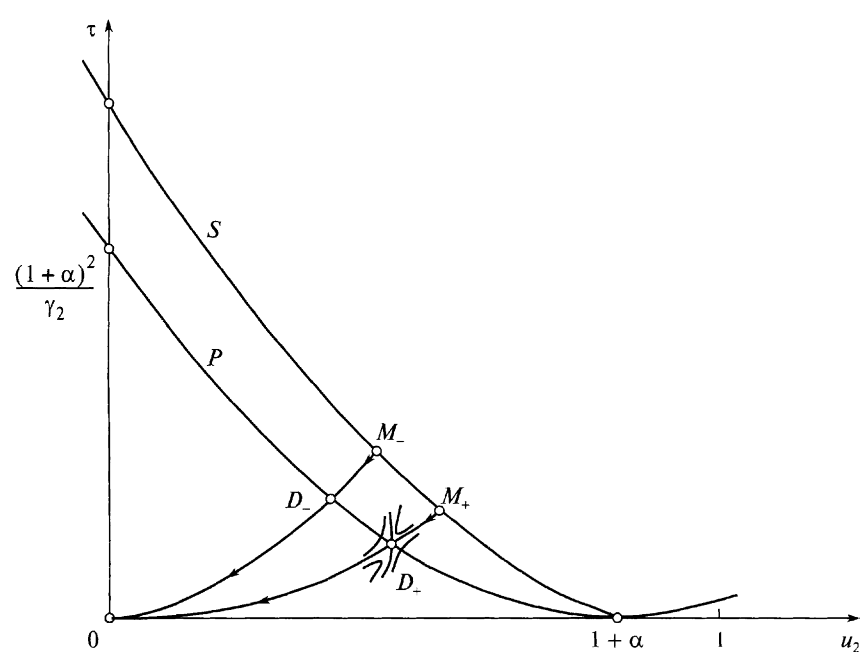

Fig. 1.

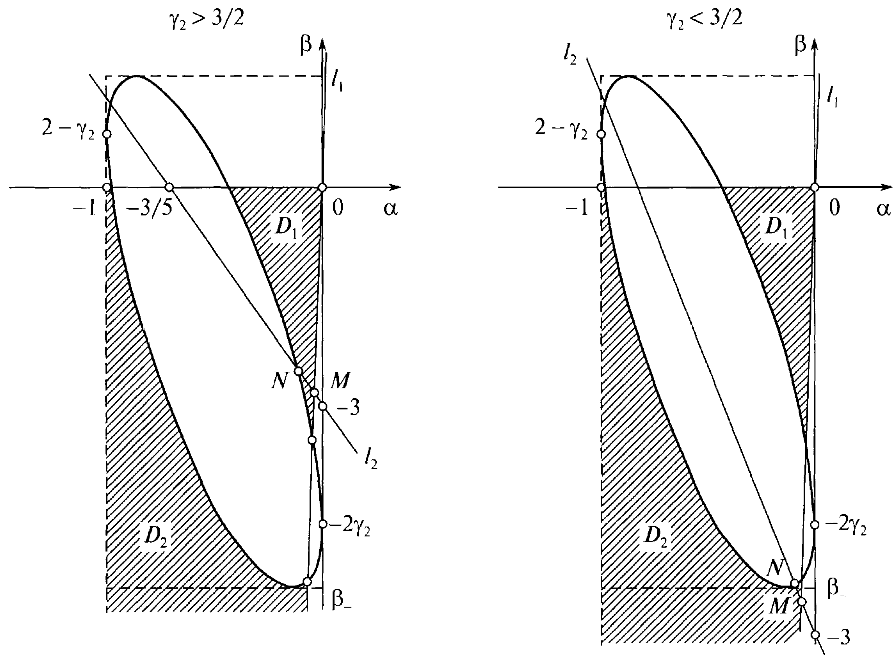

Fig. 2.

The points of intersection of the line l_1: $2\alpha-(\gamma_1-1)\,\beta=0$ with the ellipse (3.6) determine the minimum value of the parameter β in the domain D_1 and the maximum value of the parameter α in the domain D_2:

$$\beta_1=2\kappa^+,\quad \alpha_1=(\gamma_1-1)\kappa^-$$
$$\kappa^{\pm}=\gamma_2\frac{-\gamma_1-3(\gamma_1-1)\gamma_2/2\pm\sqrt{2\gamma_1\gamma_2(\gamma_1-1)}}{1+(\gamma_1-1)(\gamma_2+2)+(\gamma_1-1)^2\left(1+\gamma_2+9\gamma_2^2/4\right)} \tag{3.8}$$

The merging of the points D_+ and D_- on the parabola P corresponds to a point on the ellipse (3.6).

A further constraint, imposed on the self-similarity parameters, follows from the condition for the total energy to be finite in a sphere of radius $r_1=\xi_1|t|^{\alpha+1}$, $\xi_1>\xi_0$:

$$0<E=\int_0^{r_1}\rho\left(\varepsilon+\frac{U^2}{2}\right)r^2dr$$
$$=|t|^{\beta+5\alpha+3}\left[\int_0^{\xi_0}\lambda_1\left(\frac{K}{\gamma_1-1}+\frac{u_1^2}{2}\right)\xi^2d\xi+\int_{\xi_0}^{\xi_1}\lambda_2\left(\frac{\tau}{\gamma_2-1}+\frac{u_2^2}{2}\right)\xi^4d\xi\right]<\infty$$

when $t\to 0$. The self-similarity parameters must belong to the line

$$l_2:\beta+5\alpha+3=0 \tag{3.9}$$

In the case of self-similar compression, the energy density tends to infinity when $t\to 0$.

The point of intersection M of the lines l_1 and l_2 has the coordinates

$$\alpha_M=-3\frac{\gamma_1-1}{5\gamma_1-3},\quad \beta_M=-\frac{6}{5\gamma_1-3}$$

The point of intersection N of the line l_2 with the part of the ellipse (3.6) bounding the domain D_2 has the coordinate when $\gamma_2<3/2$

$$\alpha_N=\frac{-2\gamma_2^2+3\gamma_2-3+2\gamma_2\sqrt{2\gamma_2/3}}{3\gamma_2^2-2\gamma_2+3}<\alpha_M<\alpha_1 \tag{3.10}$$

The point N lies on the part of the ellipse (3.6) bounding the domain D_1 ($\gamma_2>3/2$), and it has the coordinate

$$\beta_N=\frac{\gamma_2^2-9\gamma_2+6-10\gamma_2\sqrt{2\gamma_2/3}}{3\gamma_2^2-2\gamma_2+3}>\beta_M>\beta_1 \tag{3.11}$$

Conditions (3.10) and (3.11) are the conditions for the existence of the self-similar convergence of a SW. They are satisfied in the case of special adiabatic exponents.

Calculation shows that a value of γ_1 can always be found for $1 \le \gamma_2 \le 2$, which is close to unity.
When $\gamma_2 = 3/2$, the line l_1 coincides with the β axis, that is $\gamma_1 = 1$. In this case of special adiabatic exponents, we have

$$\alpha_N = \alpha_M = 0, \quad \beta_N = \beta_M = -3$$

and Eq. (3.3) does not have singular points on the sound parabola (3.4).
In the neighbourhood of the singular points D_+ and D_-, we obtain

$$u_2 = \upsilon + u_3, \quad \tau = \frac{(\upsilon - \alpha - 1)^2}{\gamma_2} + \tau_3; \quad \frac{d\tau_3}{du_3} = \frac{a_{11}\tau_3 + a_{12}u_3}{a_{21}\tau_3 + a_{22}u_3}$$

where, by virtue of relations (3.5) and (3.9),

$$\begin{aligned}
&a_{11} = \gamma_2((\alpha+1)(2\alpha+\beta+2) - (\alpha+\beta+2)\upsilon) \\
&a_{12} = \gamma_2^{-1}(\upsilon-\alpha-1)[(\upsilon-\alpha-1)(2\gamma_2(\gamma_2+1) + (3\gamma_2-1)\beta \\
&+(\gamma_2^2+3\gamma_2-2)\alpha) - 2\gamma_2^2\alpha(\alpha+1)], \quad a_{21} = \gamma_2(3\gamma_2\upsilon + 2\alpha + \beta) \\
&a_{22} = \gamma_2((\alpha+1)(3\alpha+2) - 2(2\alpha+1)\upsilon)
\end{aligned} \tag{3.12}$$

The singular points are determined by the eigenvalues λ_1 and λ_2 of the matrix $A = ||a_{ij}||$:

$$\det A = \lambda_1\lambda_2, \quad \mathrm{tr}A = \lambda_1 + \lambda_2 = -\gamma_2(\upsilon - \alpha - 1)(5\alpha + \beta + 4)$$

and the eigenvectors

$$e_1 = \left(-a_{12}, \frac{a_{11} - a_{22}}{2} - \sqrt{\Delta}\right), \quad e_2 = \left(\frac{a_{11} - a_{22}}{2} - \sqrt{\Delta}, a_{21}\right), \quad \Delta = \frac{(\mathrm{tr}A)^2}{4} - \det A$$

If $0 < 4\det A < (\mathrm{tr}A)^2$, the singular point is a node and many integral curves exist joining the node at the point O with the nodes at the points D_+ and D_-. If $\det A < 0$, the singular point is a saddle point and a unique separatrix exists which goes from the point D_+ or D_- to the point O.

4. Matching conditions in a shock wave

The equations of the shock transition (1.3) are written in terms of the invariants of the operator Y (see relations (2.1) and (3.1))

$$\begin{aligned}
&D = -|t|^\alpha d(\xi), \quad \lambda_1(d - u_1) = \lambda_2(d - \xi u_2) \\
&K\lambda_1 + \lambda_1(d - u_1)^2 = \lambda_2\tau\xi^2 + \lambda_2(d - \xi u_2)^2 \\
&\lambda_1(d - u_1)\left((d - u_1)^2 + 2\gamma_1(\gamma_1 - 1)^{-1}K\right) + 2\sigma \\
&= \lambda_2(d - \xi u_2)\left((d - \xi u_2)^2 + 2\gamma_2(\gamma_2 - 1)^{-1}\xi^2\tau\right)
\end{aligned} \tag{4.1}$$

Suppose a SW moves according to the self-similar law $r = \xi_0|t|^{\alpha+1}$, $\xi_0 > 0$. Then, $d = \xi(\alpha+1) > 0$ in the case of a converging wave.
The equations of the self-similar submodel (2.2), (3.2), (4.1) allow of the expansions

$$\lambda_1\partial_{\lambda_1} + \sigma\partial_\sigma + \lambda_2\partial_{\lambda_2}, \quad \xi\partial_\xi + u_1\partial_{u_1} + d\partial_d + 3\sigma\partial_\sigma + 2K\partial_K$$

By means of these expansions it is possible to achieve satisfaction of the equalities $\xi_0 = 1$ and $L = 1$ in the last equation of (2.2). In expressions (4.1), λ_2 can be eliminated, and the equalities

$$\begin{aligned}
&\xi = 1, \quad \frac{\lambda_1}{\lambda_2} = \frac{\chi_2}{\chi_1}; \quad \chi_j = 1 + \alpha - u_j, \quad j = 1, 2 \\
&\frac{\tau}{\chi_2} + \chi_2 = \frac{K}{\chi_1} + \chi_1 \\
&\chi_2^2 + \frac{2\gamma_2}{\gamma_2 - 1}\tau = \chi_1^2 + \frac{2\gamma_1}{\gamma_1 - 1}K + \frac{2\sigma}{\lambda_1\chi_1}
\end{aligned} \tag{4.2}$$

are then satisfied in the SW. The expressions on the right-hand sides of the last two equalities on the integral curve of system (2.2) depend on the parameter K, and equalities (4.2) therefore give the curve S of the shock transitions in the plane of the variables u_2, τ, which lies above the parabola P (Fig. 1).
We now introduce the notation

$$\Lambda = \frac{\lambda_1}{\lambda_2}, \quad \kappa_j = \frac{\gamma_j - 1}{\gamma_j + 1}, \quad j = 1, 2$$

If u_2 is eliminated in system (4.1), the equation of the shock adiabatic curve is obtained

$$u_2 = u_1 + (1+\alpha - u_1)(1-\Lambda) \tag{4.3}$$

$$\frac{2\sigma}{\lambda_1}(1-\Lambda)^{1/2}\left(\frac{\tau}{\Lambda}-K\right)^{-1/2} + \tau\left(\frac{1}{\Lambda}-\frac{1}{\kappa_2}\right) - K\left(\Lambda - \frac{1}{\kappa_1}\right) = 0 \tag{4.4}$$

When $\Lambda \to \kappa_2$, we obtain the limiting compression of a strong SW $\tau \to \infty$ from the equality (4.4)

$$\Lambda \sim \kappa_2 + K\tau^{-1}\kappa_2^2\left(\frac{1}{\kappa_1}-\kappa_2\right) + \sqrt{2}\tau^{-3/2}\frac{2\sigma\kappa_2^3}{\lambda_1(\gamma_2-1)^{1/2}} + \dots \tag{4.5}$$

Substitution of the approximate expression (4.5) into Eq. (4.3) and then into system (4.2) gives a system for determining of the dependences of K and τ on the parameters α and β. If the function K is determined, the point M on the curve S is given. The integral curve joining the points O and M solves the problem. It intersects the parabola P at the point D_- in the case of a strong SW. The equality $d\xi/du_2 = 0$ is satisfied at any other point of the parabola P, and the function $\xi(u_2)$ has an extremum in contradiction to a monotonic decrease. This means that the parameters α and β must be connected by a further relation and one index α and one index β can thereby be determined in the case of a strong SW. If D_- is a node, then a set of curves joining the nodes O and D_- exists. The problem has a set of solutions. If D_- is a saddle point, then the solution is unique (Fig. 1).

If condition (3.9) is not taken into account, then, for small α and β from the domain D_1, the singular points $D_\pm$ will be nodes with a small angle between the isolated whiskers.

In this sense, the solutions are close and similar laws of convergence of a strong SW are formed.

5. Adiabatic exponents, close to special adiabatic exponents

We will assume that $\gamma_2 = 3/2 + \delta$, δ is a small quantity and condition (3.9) is satisfied. It then follows from relations (3.10) and (3.11) that $\gamma_1 = 1 + \varepsilon$, where

$$0 < \varepsilon < 4\delta^2\left(1 + 10\delta^2/27\right)/27$$

Since $\alpha_N < \alpha < \alpha_M$, $\beta_M < \beta < \beta_N$, then

$$-2\delta^2/9 < \alpha < -3\varepsilon/2, \quad 10\delta^2/9 > \beta + 3 > 15\varepsilon/2$$

We assume that $\alpha \sim -2\delta^2/9 + 2\delta^3/9$ and the close points $D_\pm$ are determined by the quantity $\upsilon_\pm \sim -\delta/3 + k_\pm\delta^2/6$ when $\delta > 0$. We also have the relations

$$a_{11} \sim -\delta/2 + (k_\pm/4 - 1/3)\delta^2, \quad a_{12} \sim -2\delta^2/9 + 2(k_\pm + 11/3)\delta^3/9$$

$$a_{21} \sim 9/4 + 15\delta/4 + (9k_\pm/8 + 1)\delta^2, \quad a_{22} \sim \delta + (-k_\pm/2 + 1/3)\delta^2$$

where $k_\pm = -1/3 \pm \sqrt{7/3}$. The asymptotic equality

$$\det A = a_{11}a_{22} - a_{12}a_{21} = \lambda_1\lambda_2 \sim -\delta^3/3 < 0$$

follows from this, and the singular points $D_\pm$ are saddle points (see Fig. 1).

Hence, there are, generally speaking two solutions of the problem of the self-similar convergence of a SW, each of which is completely determined by the initial convergence.

Remark. Behind the self-similar wave, the gas is carried along to the centre. In order to maintain self-similar convergence, the piston must compress the gas into the centre. The work of the piston in overcoming the gas pressure tends to infinity. This means that it sets in at the instant when the piston starts to move according to a non-self-similar law. The characteristic of the rarefaction wave produced from the points of the piston at this instant mounts up to a shock wave The rarefaction wave begins to arrest and attenuate the shock wave together with an increasing counterpressure in front of the wave. The instant of disappearance of the strong jump comes, converting it into a weak jump. Shock-free compression then develops which leads to the formation of a shock wave before or after reflection from the centre.[16] The reflected wave moves through the stopping converging gas to the piston giving rise to a back pulse (a recoil).

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